

GRAPH REDUCTION TECHNIQUES AND THE MULTIPLICITY OF THE LAPLACIAN EIGENVALUES

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ABSTRACT. Let $M = [m_{ij}]$ be an $n \times m$ real matrix, ρ be a nonzero real number, and A be a symmetric real matrix. We denote by $D(M)$ the $n \times n$ diagonal matrix $\text{diag}(\sum_{j=1}^m m_{1j}, \dots, \sum_{j=1}^m m_{nj})$ and denote by L_A^ρ the generalized Laplacian matrix $D(A) - \rho A$. A well-known result of Grone et al. states that by connecting one of the end-vertices of P_3 to an arbitrary vertex of a graph, does not change the multiplicity of Laplacian eigenvalue 1. We extend this theorem and some other results for a given generalized Laplacian eigenvalue μ . Furthermore, we give two proofs for a conjecture by Saito and Woei on the relation between the multiplicity of some Laplacian eigenvalues and pendant paths.

1. INTRODUCTION

Let A be a real symmetric matrix. There exists a unique weighted graph G such that the adjacency matrix of G , is A ; i.e. for $i \neq j$, A_{ij} is the weight of the edge $\{i, j\}$ and A_{ii} is twice of the weight of the loop at the vertex i . In this paper, we look at real symmetric matrices in this point of view.

For a positive integer n , we denote by $\text{Sym}_n(\mathbb{R})$, the set of real symmetric matrices of order n and denote by $[n]$ the set $\{1, \dots, n\}$. The multiplicity of an eigenvalue λ of A is denoted by $m_A(\lambda)$ and a λ -eigenvector of A is an eigenvector of A corresponding to λ . For two positive integers i and n , \mathbf{j}_n and \mathbf{e}_i , denote the all 1's vector and the vector with a 1 in the i^{th} coordinate and 0's elsewhere, respectively, in \mathbb{R}^n . The restriction of a vector \mathbf{x} to any index set I is denoted by \mathbf{x}_I and we denote the entry of \mathbf{x} corresponding to an index u , by $\mathbf{x}(u)$. The identity matrix is denoted by \mathbb{I}_n or briefly \mathbb{I} . The path, the cycle, and the star graph on n vertices are denoted by P_n , C_n , and S_n , respectively.

Let $M = [m_{ij}]$ be an $n \times m$ real matrix. The transpose of M is denoted by M^T and we denote by $D(M)$, the $n \times n$ diagonal matrix $\text{diag}(\sum_{j=1}^m m_{1j}, \dots, \sum_{j=1}^m m_{nj})$. For $\rho \in \mathbb{R} - \{0\}$, we denote by L_A^ρ , the *generalized Laplacian matrix* $D(A) - \rho A$. If $\rho = 1$ ($\rho = -1$) and $A(G)$ is the adjacency matrix of a given graph G , then we have the Laplacian matrix $L_{A(G)}^1 = L(G)$ (signless Laplacian matrix $L_{A(G)}^{-1} = Q(G)$, respectively).

Let μ be a Laplacian eigenvalue of G . We consider the results about the relation between $m_{L(G)}(\mu)$ and $m_{L(H)}(\mu)$, for a particular subgraph H of G . We recall some of these results:

Theorem 1. [6] *If G' is a graph obtained from G by connecting one of the end-vertices of P_3 to an arbitrary vertex of G , then we have $m_{L(G)}(1) = m_{L(G')}(1)$.*

Theorem 2. [8] *Let G be any graph with a simple Laplacian eigenvalue μ . Let u be a vertex of G such that an eigenvector corresponding to μ is nonzero on u . Let H be any graph, and let G' be the graph formed by joining an arbitrary vertex of H to u . Then $m_{L(H)}(\mu) = m_{L(G')}(1)$.*

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A connected sum of two graphs G_1 and G_2 is any graph G where $V(G) = V(G_1) \cup V(G_2)$ and $E(G)$ differs from $E(G_1) \cup E(G_2)$ by the addition of a single edge joining some (arbitrary) vertex of $V(G_1)$ to some vertex of $V(G_2)$, and is denoted by $G = G_1 \# G_2$ [6].

Theorem 3. [6] *Let G be a nonempty graph on n vertices. Let $H = G \# S_k$ be a connected sum of G with the star on $k > 1$ vertices. Then $m_{L(G)}(k) = m_{L(H)}(k)$*

A *cluster* of a graph G is an independent set of two or more vertices of G , each of which has the same set of neighbours. The degree of a cluster is the cardinality of its shared set of neighbours, i.e., the common degree of each vertex in the cluster. A d -cluster is a cluster of degree d . The number of vertices in a d -cluster is its order. A collection of two or more d -clusters is independent if the sets of vertices comprising the d -clusters are pairwise disjoint [5].

Theorem 4. [5] *Let G be a graph with k independent d -clusters of orders r_1, \dots, r_k . Then $m_{L(G)}(d) \geq \sum_{i=1}^k r_i - k$.*

Among other results, we generalize these results for weighted graphs and an arbitrary generalized Laplacian eigenvalue μ .

A *pendant path* of a graph G is a path such that one of its end vertices has degree one and all the internal vertices have degree two and other end vertex has degree greater than two. $p_k(G)$ denotes the number of pendant paths of length k , and $q_k(G)$ is the number of vertices with degree greater than three which are an end vertex of some pendant paths of length k . If $k = 1$, we have the well-known result of Faria [3] that $m_{L(G)}(1) \geq p_1(G) - q_1(G)$. Saito and Woei [9] conjectured that for any positive integer k , any graph G has some Laplacian eigenvalue with multiplicity at least $p_k(G) - q_k(G)$ and proved it for $k = 2$. The following generalization of the conjecture has been proved in [4]. We give two proofs for this theorem in the next sections.

Theorem 5. [4] *Let G be a graph. Then $4\cos^2(\frac{\pi i}{2k+1})$ for any $k \geq 1$ and $i = 1, \dots, k$, is both a Laplacian and a signless Laplacian eigenvalue of G with multiplicity at least $p_k(G) - q_k(G)$.*

Let $A \in \text{Sym}_n(\mathbb{R})$ and λ be an eigenvalue of A of multiplicity k . A set $U \subseteq [n]$ is a *star set* for λ (or λ -star set) of A if $|U| = k$ and λ is not an eigenvalue of the submatrix of A obtained by removing rows and columns with index in U . It is known that for every eigenvalue λ there exists a λ -star set [2].

We recall the following theorem about star sets that we use in the next sections.

Theorem 6. [2, Theorem 7.2.6] *Let U be a λ -star set of A . If $m_A(\lambda) = k$, then there exists a basis of eigenvectors $\{\alpha_s : s \in U\}$ such that $\alpha_s(t) = \delta_{st}$, whenever $s, t \in U$ and δ is the Kronecker delta function; its value is 1 if $s = t$, and 0 otherwise.*

2. TYPE I REDUCTIONS: EDGE DELETING

In this section, for a given eigenvalue μ , we remove a particular subgraph corresponding to μ and consider the multiplicity of μ of remaining graph.

First, we state this following Edge Principle Theorem.

Theorem 7. [7] *Let μ be a Laplacian eigenvalue of G afforded by eigenvector \mathbf{x} . If $x_i = x_j$, then μ is an eigenvalue of G' afforded by \mathbf{x} , where G' is the graph obtained from G by deleting or adding $e = \{i, j\}$ depending on whether or not it is an edge of G .*

Now, we state a weighted version of theorem above, for the Laplacian and the signless Laplacian of weighted graphs:

Lemma 8. *Let $n \in \mathbb{N}$, $\rho \in \{-1, 1\}$, $A \in \text{Sym}_n(\mathbb{R})$, and μ be an eigenvalue of L_A^ρ with a μ -eigenvector \mathbf{x} . Suppose that $a \in \mathbb{R}$ and $x_i = \rho x_j$, for some $i, j \in [n]$, $i \neq j$. If A' is the matrix obtained from A by setting $A'_{ij} = A'_{ji} = A_{ij} + a$, then \mathbf{x} is a μ -eigenvector of $L_{A'}^\rho$.*

Proof. We have $L_A^\rho = L_{A'}^\rho - \begin{matrix} i & j \\ \begin{pmatrix} a & -\rho a \\ -\rho a & a \end{pmatrix} \end{matrix}$. So,

$$\mu \mathbf{x} = L_A^\rho \mathbf{x} = L_{A'}^\rho \mathbf{x} - \begin{matrix} i & j \\ \begin{pmatrix} a & -\rho a \\ -\rho a & a \end{pmatrix} \end{matrix} \mathbf{x} = L_{A'}^\rho \mathbf{x} - \begin{matrix} i & j \\ \begin{pmatrix} 0 & ax_i - \rho ax_j \\ -\rho ax_i + ax_j & 0 \end{pmatrix} \end{matrix} = L_{A'}^\rho \mathbf{x}. \quad \square$$

The following theorem is the main theorem of this section.

Theorem 9. *Let $\mu \in \mathbb{R}$, $\rho \in \{-1, 1\}$, and \mathcal{H}, \mathcal{L} be real symmetric matrices with row and column indices $I = I_1 \dot{\cup} I_2 \dot{\cup} I_3$ and $J = J_1 \dot{\cup} J_2$, respectively. Suppose that $I_1 \dot{\cup} I_2$ is a μ -star set of $L_{\mathcal{H}}^\rho$. If X, \mathcal{G} , and \mathcal{E} are matrices given below,*

$$X = \begin{matrix} I_{11} & I_{12} \\ \begin{pmatrix} \mathbf{x}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{x}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{x}_{|J_1|} \end{pmatrix} \end{matrix}, \mathcal{G} = \begin{matrix} I_1 & I_2 & I_3 & J_1 & J_2 \\ \begin{pmatrix} I_1 & & & X & \\ I_2 & & \mathcal{H} & 0 & \mathcal{A} \\ I_3 & & & 0 & \\ J_1 & X^T & 0 & 0 & \\ J_2 & & \mathcal{A}^T & & \mathcal{L} \end{pmatrix} \end{matrix}, \mathcal{E} = \begin{matrix} I_1 & I_2 & I_3 & J_1 & J_2 \\ \begin{pmatrix} I_1 & & & 0 & \\ I_2 & & \mathcal{H} & 0 & \mathcal{A} \\ I_3 & & & 0 & \\ J_1 & 0 & 0 & 0 & \\ J_2 & & \mathcal{A}^T & & \mathcal{L} \end{pmatrix} \end{matrix},$$

for nowhere-zero vectors $\{\mathbf{x}_i\}$ and a matrix \mathcal{A} , where $D(\mathcal{A}) = 0$ and $\mathcal{A}^T \boldsymbol{\alpha} = \mathbf{0}$, for every μ -eigenvector $\boldsymbol{\alpha}$ of $L_{\mathcal{H}}^\rho$, then $m_{L_{\mathcal{E}}^\rho}(\mu) = m_{L_{\mathcal{G}}^\rho}(\mu) + |I_1|$.

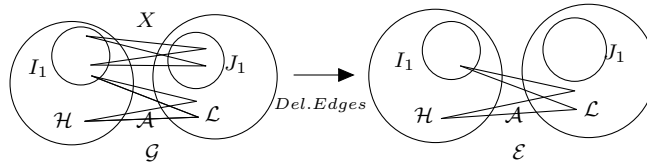
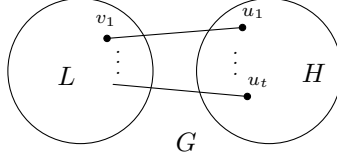


FIGURE 1. The graphs of Theorem 9.

In Theorem 9, by putting $\mathcal{A} = 0$, we conclude the following corollary for the (signless) Laplacian matrix of simple graphs:

Corollary 10. *Let $\mu \in \mathbb{R}$, $\rho \in \{-1, 1\}$, and H be a graph and $\{u_1, \dots, u_t\}$ be a subset of a μ -star set of L_H^ρ . If L is an arbitrary graph disjoint from H , and G is the graph formed by joining the vertex u_i to an arbitrary vertex v_i of L (not necessarily disjoint), $i \in [t]$, then $m_{L_G^\rho}(\mu) = m_{L_L^\rho}(\mu) + m_{L_H^\rho}(\mu) - t$.*

FIGURE 2. Graph G of Corollary 10.

As a particular case of Corollary 10 ($t = 1$), we have this theorem of [8]:

Corollary 11. [8] *Let G be any graph with a simple Laplacian eigenvalue μ . Let u be a vertex of G such that an eigenvector corresponding to μ is nonzero on u . Let H be any graph, and let G' be the graph formed by joining an arbitrary vertex of H to u . Then $m_{L(H)}(\mu) = m_{L(G')}(\mu)$.*

Proof. Assume that $L_G^\rho = \frac{u}{v(G) - \{u\}} \left(\begin{array}{c|c} a & \mathbf{x}^T \\ \hline \mathbf{x} & M \end{array} \right)$ and α is a μ -eigenvector of L_G^ρ such that $\alpha(u) \neq 0$. It is sufficient to show that $\{u\}$ is a μ -star set of L_G^ρ . On the other hand, we show $m_M(\mu) = 0$. Suppose, by contradiction, M has a μ -eigenvector β . If $\mathbf{x}^T \beta = 0$, then \mathbf{y} is a μ -eigenvector of L_G^ρ , where $\mathbf{y}(v) = \begin{cases} \beta(v) & v \neq u, \\ 0 & v = u \end{cases}$. Since $\alpha(u) \neq 0$, the vectors α and \mathbf{y} are independent and we have a contradiction with $m_{L_G^\rho}(\mu) = 1$. If $\mathbf{x}^T \beta \neq 0$, then $0 = (\mu \mathbb{I} - L_G^\rho) \alpha \Rightarrow \alpha(u) \mathbf{x} = (\mu \mathbb{I} - M) \alpha_{|V(G) - \{u\}} \Rightarrow \alpha(u) \beta^T \mathbf{x} = \beta^T (\mu \mathbb{I} - M) \alpha_{|V(G) - \{u\}} = 0 \xrightarrow{\beta^T \mathbf{x} \neq 0} \alpha(u) = 0$, and we have a contradiction. This completes the proof. \square

Remark 12. By Corollary 10, since $m_{L(P_3)}(1) = 1$ and the value of a 1-eigenvector is nonzero on every pendant vertex of P_3 , we have Theorem 1. Also, $m_{L(S_k)}(k) = 1$ and every k -eigenvector of S_k is nowhere-zero, hence we have Theorem 3.

2.1. Edge Switching. In the following theorem, for a given eigenvalue μ , a particular subgraph, and given weights of the edges, we delete some edges and switch some weights from a section of graph to another section and give the relation between the multiplicity of μ for two graphs.

Theorem 13. *Let $\mu \in \mathbb{R}$, $\rho \in \mathbb{R} - \{0\}$ and \mathcal{H}, \mathcal{L} be real symmetric matrices with row and column indices $I = I_1 \dot{\cup} I_2 \dot{\cup} I_3$ and $J = J_1 \dot{\cup} J_2$, respectively. Suppose that $S \in \text{Sym}_{|I_1|}(\mathbb{R})$ and $I_1 \dot{\cup} I_2$ is a μ -star set of $L_{\mathcal{H}}^\rho$. If $\hat{\mathcal{H}}, \hat{\mathcal{L}}, \mathcal{G}$, and \mathcal{E} are symmetric matrices given below,*

$$\hat{\mathcal{H}} = \mathcal{H} + \left(\begin{array}{c|c} I_1 & I \setminus I_1 \\ \hline S & 0 \\ 0 & 0 \end{array} \right), \quad \hat{\mathcal{L}} = \mathcal{L} - \left(\begin{array}{c|c} J_1 & J_2 \\ \hline S' & 0 \\ 0 & 0 \end{array} \right), \quad \mathcal{G} = \frac{\begin{array}{ccc|cc} I_1 & I_2 & I_3 & J_1 & J_2 \\ \hline I_1 & & & X & \\ I_2 & & \hat{\mathcal{H}} & 0 & \mathcal{A} \\ I_3 & & & 0 & \\ \hline J_1 & X^T & 0 & 0 & \\ J_2 & \mathcal{A}^T & & & \mathcal{L} \end{array}}{\begin{array}{ccc|cc} I_1 & I_2 & I_3 & J_1 & J_2 \\ \hline I_1 & & & 0 & \\ I_2 & & \mathcal{H} & 0 & \mathcal{A} \\ I_3 & & & 0 & \\ \hline J_1 & 0 & 0 & 0 & \\ J_2 & \mathcal{A}^T & & & \hat{\mathcal{L}} \end{array}}, \quad \mathcal{E} = \frac{\begin{array}{ccc|cc} I_1 & I_2 & I_3 & J_1 & J_2 \\ \hline I_1 & & & 0 & \\ I_2 & & \mathcal{H} & 0 & \mathcal{A} \\ I_3 & & & 0 & \\ \hline J_1 & 0 & 0 & 0 & \\ J_2 & \mathcal{A}^T & & & \hat{\mathcal{L}} \end{array}}{\begin{array}{ccc|cc} I_1 & I_2 & I_3 & J_1 & J_2 \\ \hline I_1 & & & 0 & \\ I_2 & & \mathcal{H} & 0 & \mathcal{A} \\ I_3 & & & 0 & \\ \hline J_1 & 0 & 0 & 0 & \\ J_2 & \mathcal{A}^T & & & \hat{\mathcal{L}} \end{array}},$$

for some matrices \mathcal{A} and X , where $D(\mathcal{A}) = 0$, $\mathcal{A}^T \alpha = \mathbf{0}$, for every μ -eigenvector α of \mathcal{H} , and S' is a solution of the equation $L_{S'}^\rho + D(X^T) = \rho^2 X^T (L_S^\rho + D(X))^{-1} X$, then $m_{L_{\mathcal{E}}^\rho}(\mu) = m_{L_{\mathcal{G}}^\rho}(\mu) + |I_1|$.

In particular, if L_S^ρ is invertible, then $m_{L_{\mathcal{H}}^\rho}(\mu) = m_{L_{\mathcal{G}}^\rho}(\mu) - |I_1|$.

For any $\rho \in \mathbb{R} \setminus \{0, 1\}$ and $L \in \text{Sym}_n(\mathbb{R})$, it is easy to see that the equation $L = L_M^\rho$ has a unique solution $M \in \text{Sym}_n(\mathbb{R})$. Thus, for given S and X such that there exists $(L_S^\rho + D(X))^{-1}$, the equation $L_{S'}^\rho + D(X^T) = \rho^2 X^T (L_S^\rho + D(X))^{-1} X$ has a solution for S' .

Corollary 14 (Edge Switching). *Let $\rho^2 = 1$ and L, H be two disjoint graphs. With the notations of Theorem 13, put $\mathcal{H} = A(H)$, $\mathcal{L} = A(L)$, $\mathcal{A} = 0$, and Consider the following two cases:*

- (1) $X = \mathbb{I}_{|I_1|}$ and $-S$ is a permutation matrix corresponding to an involution,
- (2) for a given S , suppose that X is a solution of $X = L_S^\rho + D(X)$.

Then, for both cases, $S' = S$ is a solution and $m_{L_G^\rho}(\mu) + |I_1| = m_{L_H^\rho}(\mu) + m_{L_{\hat{L}}^\rho}(\mu)$.

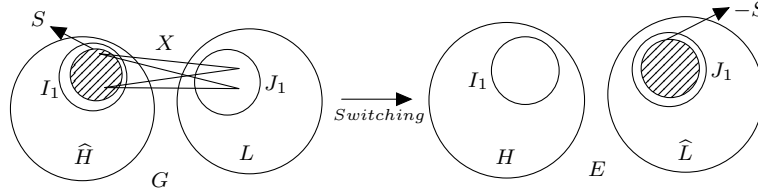


FIGURE 3. A schematic diagram of graphs G and E of Corollary 14.

Since for a non-bipartite graph H , the signless Laplacian matrix $Q(H)$ is invertible, by a particular case of Theorem 13, if we set $S = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then we can conclude the next corollary.

Corollary 15. *Let H be a non-bipartite graph with a μ -star set U , for a signless Laplacian eigenvalue μ . If $u, v \in U$ and $uv \in E(H)$ ($uv \notin E(H)$), then removing (adding, resp.) the edge uv , decreases $m_{Q(H)}(\mu)$ by 2 and $U \setminus \{u, v\}$ is a μ -star set of $Q(\hat{H})$.*

Suppose that G is an r -regular graph and $\lambda \in \mathbb{R}$. If $m_{A(G)}(\lambda) = k$ and U is a λ -star set of $A(G)$, then U is an $(r - \rho\lambda)$ -star set of L_G^ρ . So, we can conclude some results of Laplacian matrices similar to adjacency matrices of regular subgraphs. For example, we state the following corollary on path subgraphs for Laplacian matrices such that there is in [1] for adjacency matrices of graphs:

Corollary 16. *A path with n vertices of valency 2 in a graph G can be replaced by an edge (see Figure 4) without changing the multiplicity of Laplacian eigenvalue $4\sin^2(\frac{k\pi}{n})$ (signless Laplacian eigenvalue $4\cos^2(\frac{k\pi}{n})$), for $n \geq 3$ and $k \in [n-1], k \neq \frac{n}{2}$.*

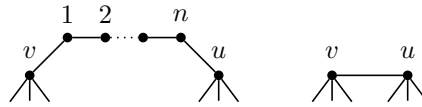


FIGURE 4. The graphs of Corollary 16.

Proof. We prove the corollary for $\rho = 1$, the other case is similar. The eigenvalues of $L_{C_n}^1$ are $(4\sin^2(\frac{k\pi}{n}))_{k=0}^{n-1}$ (see [10]). If $\mu = 4\sin^2(\frac{k\pi}{n})$ for an integer k , $k \in [n-1]$ and $k \neq \frac{n}{2}$, then $m_{L_{C_n}^1}(\mu) = 2$. Since every set of two adjacent vertices of C_n is a $(2 - \mu)$ -star set of $A(C_n)$ [1], so it is a μ -star set of $L_{C_n}^1$. Thus, by Corollary 14, the proof is complete. \square

Example 17. In Corollary 16:

- $n = 3$ and $k = 1$: A path with three vertices of valency 2 in a graph G can be replaced by an edge, without changing the Laplacian multiplicity of 3 and the signless Laplacian multiplicity of 1.
- $n = 4$ and $k = 1$: A path with four vertices of valency 2 in a graph G can be replaced by an edge, without changing the (signless) Laplacian multiplicity of 2.

3. TYPE II REDUCTIONS: DELETING SUBGRAPHS

In this section we generalize Theorem 4 for a given generalized Laplacian eigenvalue μ .

Theorem 18. *Let $\mu \in \mathbb{R}, \rho \in \mathbb{R} - \{0\}, r \in \mathbb{N}$, and $\mathcal{H}, \{\mathcal{H}_i\}_{i=1}^{r-1}, \mathcal{L}, \{\mathcal{L}_i\}_{i=1}^{r-1}, \mathcal{K}$ be symmetric matrices. Suppose that $L_{\mathcal{E}_i}^\rho$ has a μ -eigenvector γ^i such that $\gamma^i|_J = \mathbf{0}$ and $\gamma^i|_{I_i} \neq \mathbf{0}$, $i \in [r-1]$, and $L_{\mathcal{E}}^\rho$ has independent μ -eigenvectors β^i such that $\beta^i|_J = \mathbf{0}$, $i \in [s]$. Then $m_{L_G^\rho}(\mu) \geq s + r - 1$, where*

$$\mathcal{E}_i = \begin{array}{c|c|c|c} I & \mathcal{H} & \mathcal{A} & 0 \\ \hline J & \mathcal{A}^T & \mathcal{L}_i & B_i \\ \hline I_i & 0 & B_i^T & \mathcal{H}_i \end{array}, \quad i \in [r-1], \quad \mathcal{E} = \begin{array}{c|c|c|c} \mathcal{H} & \mathcal{A} & 0 \\ \hline \mathcal{A}^T & \mathcal{L} & B \\ \hline 0 & B^T & \mathcal{K} \end{array}, \quad \mathcal{G} = \begin{array}{c|c|c|c|c|c|c} I & \mathcal{H} & 0 & \cdots & 0 & \mathcal{A} & 0 \\ \hline I_1 & 0 & \mathcal{H}_1 & \cdots & 0 & B_1 & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline I_{r-1} & 0 & 0 & \cdots & \mathcal{H}_{r-1} & B_{r-1} & 0 \\ \hline J & \mathcal{A}^T & B_1^T & \cdots & B_{r-1}^T & \mathcal{L} & B \\ \hline J_1 & 0 & 0 & \cdots & 0 & B^T & \mathcal{K} \end{array}.$$

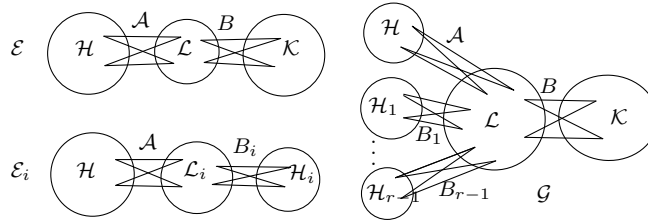


FIGURE 5. A schematic figure of Theorem 18.

Now, we can conclude the following theorem of [5] on d -clusters.

Corollary 19. [5] *Let G be a graph with k independent d -clusters of orders r_1, \dots, r_k . Then $m_{L(G)}(d) \geq \sum_{i=1}^k r_i - k$.*

Proof. With the notations of Theorem 18, we put $\rho = 1, \mathcal{H} = \mathcal{H}_i = [0]_{1 \times 1}, \mathcal{L}_i = [0]_{d \times d}$, and $\mathcal{A} = B_i^T = \mathbf{j}_d$, then $m_{L_{\mathcal{E}_i}^\rho}(d) = 1$, for $d \neq 2$, and $m_{L_{\mathcal{E}_i}^\rho}(d) = 2$, for $d = 2$, and $\gamma^i = (1, 0, \dots, 0, -1)^T$ is a d -eigenvector. Since d -clusters are independent, by using Theorem 18, k times, we have $m_{L_G^\rho}(d) \geq \sum_{i=1}^k (r_i - 1) = \sum_{i=1}^k r_i - k$. \square

Now, we give our first proof for Theorem 5. First, we need the following lemma on eigenvectors of the path graph.

Lemma 20. [10] *Let n be a positive integer. Then $4 \cos^2(\frac{j\pi}{2n})$ for $j \in [n]$ ($4 \sin^2(\frac{l\pi}{2n})$ for $0 \leq l \leq n-1$) is a Laplacian eigenvalue of P_n with the corresponding eigenvector \mathbf{v}_j , where $\mathbf{v}_j(u) = \cos(\frac{(n-j)(2u-1)\pi}{2n})$, for $u \in [n]$.*

Since the signless Laplacian matrix and the Laplacian matrix of a path are similar, it is easy to see that \mathbf{w}_j is a signless Laplacian eigenvector corresponding to $4 \cos^2(\frac{j\pi}{2n})$, where $\mathbf{w}_j(u) = (-1)^u \mathbf{v}_j(u)$, for $j, u \in [n]$.

First proof of Theorem 5: By Lemma 20, if $n = 2k + 1$, $j = 2t$, and $u = k + 1$, then $\mathbf{v}_j(u) = 0$. With the notations of Theorem 18, we put $\mu = 4 \cos^2(\frac{2t\pi}{2(2k+1)})$, $\rho = 1$, $\mathcal{H} = \mathcal{H}_i = A(P_k)$, $\mathcal{L}_i = [0]_{1 \times 1}$, $\mathcal{A} = \mathbf{e}_k$, and $B_i^T = \mathbf{e}_1$, then we have $\mathcal{E}_i = A(P_{2k+1})$. If we put $\boldsymbol{\gamma}^i = \mathbf{v}_j$, then by using Theorem 18, $q_k(G)$ times, we have $m_{L_G^\rho}(4 \cos^2(\frac{t\pi}{2k+1})) \geq p_k(G) - q_k(G)$, $t \in [k]$. By similar proof, we have the statement is true for $\rho = -1$.

4. TYPE III REDUCTIONS: SPLITTING VERTICES

In this section, we state a splitting method to simplify graphs for a generalized Laplacian eigenvalue μ and a particular subgraph corresponding to it.

Theorem 21. Let $\mu \in \mathbb{R}$, $\rho \in \mathbb{R} - \{0\}$ and \mathcal{H}, \mathcal{L} be real symmetric matrices. Suppose that $m_{L_{\mathcal{H}+D(\mathbf{x})}^\rho}(\mu) = 1$. If $\mathbf{x}^T \boldsymbol{\alpha} \neq 0$, for a μ -eigenvector $\boldsymbol{\alpha}$ of $L_{\mathcal{H}}^\rho + D(\mathbf{x})$, then $m_{L_{\mathcal{E}}}^\rho(\mu) = m_{L_{\mathcal{E}}}^\rho(\mu)$, where

$$G = \frac{I}{J} \left(\begin{array}{c|c|c|c} \mathcal{H} & \mathbf{x} & 0 \\ \hline v & \mathbf{x}^T & a & \mathbf{y}^T \\ \hline 0 & \mathbf{y} & \mathcal{L} \end{array} \right), \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{|J|} \end{pmatrix}, \quad \mathcal{E} = \begin{array}{c|c|c|c|c|c} I_1 & \left(\begin{array}{c|c} \mathcal{H} & \mathbf{x} \\ \hline \mathbf{x}^T & a \end{array} \right) & 0 & \cdots & 0 & \frac{0}{y_1 \mathbf{e}_1^T} \\ \hline I_2 & 0 & \left(\begin{array}{c|c} \mathcal{H} & \mathbf{x} \\ \hline \mathbf{x}^T & a \end{array} \right) & \cdots & 0 & \frac{0}{y_2 \mathbf{e}_2^T} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline I_{|J|} & 0 & 0 & \cdots & \left(\begin{array}{c|c} \mathcal{H} & \mathbf{x} \\ \hline \mathbf{x}^T & a \end{array} \right) & \frac{0}{y_{|J|} \mathbf{e}_{|J|}^T} \\ \hline J & 0 & y_1 \mathbf{e}_1 & y_2 \mathbf{e}_2 & \cdots & y_{|J|} \mathbf{e}_{|J|} \\ \hline & & & & & \mathcal{L} \end{array}$$

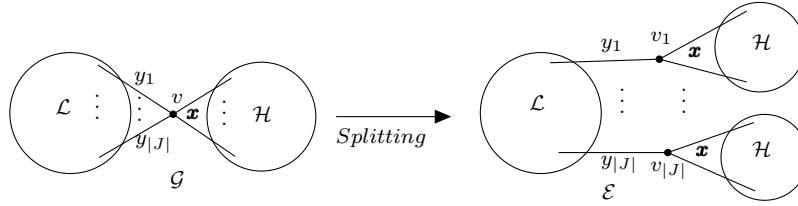


FIGURE 6. The splitting method: Theorem 21.

Lemma 22. Let $n \in \mathbb{N}$, $a, \mu \in \mathbb{R}$, $\rho \in \mathbb{R} - \{0\}$, $\mathcal{H} \in \text{Sym}_n(\mathbb{R})$, and $\mathbf{x} \in \mathbb{R}^n$. The following statements are equivalent:

- (i) $m_{L_{\mathcal{H}+D(\mathbf{x})}^\rho}(\mu) = 1$ and $\mathbf{x}^T \boldsymbol{\alpha} \neq 0$, for a μ -eigenvector $\boldsymbol{\alpha}$ of $L_{\mathcal{H}}^\rho + D(\mathbf{x})$;
- (ii) $m_{L_{\mathcal{K}}}^\rho(\mu) = 1$ and $\boldsymbol{\beta}(v) = 0$, for a μ -eigenvector $\boldsymbol{\beta}$ of $L_{\mathcal{K}}^\rho$.

Furthermore, (i) and (ii) imply $m_{L_{\widehat{\mathcal{H}}}}^\rho(\mu) = 0$, where

$$\mathcal{K} = \frac{I}{I'} \left(\begin{array}{c|c|c|c} \mathcal{H} & \mathbf{x} & 0 \\ \hline v & \mathbf{x}^T & a & \mathbf{x}^T \\ \hline 0 & \mathbf{x} & \mathcal{H} \end{array} \right) \quad \text{and} \quad \widehat{\mathcal{H}} = \frac{I}{v} \left(\begin{array}{c|c} \mathcal{H} & \mathbf{x} \\ \hline \mathbf{x}^T & a \end{array} \right).$$

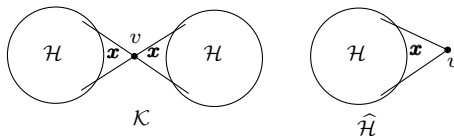


FIGURE 7. The graphs of Lemma 22.

Proof. Suppose that α, β, γ are μ -eigenvectors of $L_{\mathcal{H}}^{\rho} + D(\mathbf{x})$, $L_{\mathcal{K}}^{\rho}$, and $L_{\widehat{\mathcal{H}}}^{\rho}$, respectively.

(i) \Rightarrow (ii): For the index I of $L_{\mathcal{K}}^{\rho}$, we have

$$(1) \quad (\mu \mathbb{I} - L_{\mathcal{H}}^{\rho} - D(\mathbf{x}))\beta_{|I} = -\rho \begin{pmatrix} \mathbf{x} & | & 0 \end{pmatrix} \begin{pmatrix} \beta(v) \\ \beta_{|I'} \end{pmatrix}.$$

Multiplying relation (1) by α^T from the left, we obtain

$$0 = \alpha^T (\mu \mathbb{I} - L_{\mathcal{H}}^{\rho} - D(\mathbf{x}))\beta_{|I} = -\rho \alpha^T \mathbf{x} \beta(v).$$

Hence, $\beta(v) = 0$ and $\beta_{|I} = a_1 \alpha$ and similarly $\beta_{|I'} = a_2 \alpha$, for some $a_1, a_2 \in \mathbb{R}$. For the index v of $L_{\mathcal{K}}^{\rho}$, we have

$$(\mu - a + \rho a - 2D(\mathbf{x}^T))\beta(v) = -\rho(\mathbf{x}^T \beta_{|I} + \mathbf{x}^T \beta_{|I'}) = -\rho(a_1 + a_2)\mathbf{x}^T \alpha.$$

Hence $a_2 = -a_1$ and the proof is done.

(ii) \Rightarrow (i): It follows by the relations above in a similar manner.

Now, we show that $m_{L_{\widehat{\mathcal{H}}}^{\rho}}(\mu) = 0$. We have

$$(2) \quad (\mu \mathbb{I} - L_{\widehat{\mathcal{H}}}^{\rho} - D(\mathbf{x}))\gamma_{|I} = -\rho \mathbf{x} \gamma(v).$$

Multiplying relation (2) by α^T from the left, we obtain $\gamma(v) = 0$ and $\gamma_{|I} = b\alpha$, for a $b \in \mathbb{R}$. For the index v of $L_{\widehat{\mathcal{H}}}^{\rho}$, we have

$$(\mu - a + \rho a - D(\mathbf{x}^T))\gamma(v) = -\rho \mathbf{x}^T \gamma_{|I} = -\rho b \mathbf{x}^T \alpha.$$

Hence $b = 0$ and $m_{L_{\widehat{\mathcal{H}}}^{\rho}}(\mu) = 0$. □

The following corollary is a straightforward consequence of Theorem 21 and Lemma 22.

Corollary 23. *Let $\mu \in \mathbb{R}, \rho \in \mathbb{R} - \{0\}$ and H, L be two disjoint graphs and $u \in V(H)$ and $v \in V(L)$. Suppose that H_1, \dots, H_t are t copies of H and E, K, G are graphs as shown below (see Figure 8). If $m_{L_K^{\rho}}(\mu) = 1$ and $\beta(v') = 0$, for a μ -eigenvector β of L_K^{ρ} , then $m_{L_E^{\rho}}(\mu) = m_{L_G^{\rho}}(\mu) + t - 1$.*

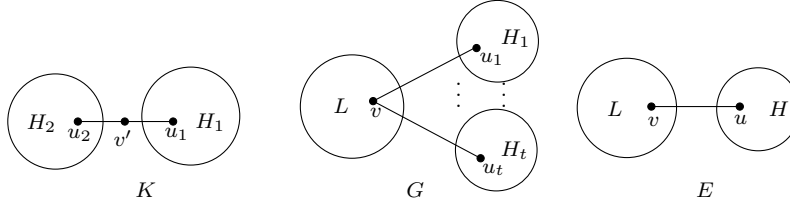


FIGURE 8. The graphs of Corollary 23.

Now, we give the second proof for Theorem 5.

Second proof of Theorem 5: For brevity, we set $q_k(G) = q$, $p_k(G) = p$. With the notations of Corollary 23, we put $\mu = 4 \cos^2(\frac{2t\pi}{2(2k+1)})$, $\rho = \pm 1$, $H = P_k$, then by using the splitting method of Corollary 23, for q vertices of G , q times, we have $m_{L_G^{\rho}}(4 \cos^2(\frac{t\pi}{2k+1})) = m_{L_E^{\rho}}(4 \cos^2(\frac{t\pi}{2k+1})) + p - q \geq p - q$, for $t \in [k]$.

Example 24. $\mu = 4 \cos^2(\frac{t\pi}{2k+1})$: Suppose that $k \in \mathbb{N}$ and G, E are the graphs as shown below (see Figure 9). Then $m_{L(G)}(\mu) = m_{L(E)}(\mu)$ and $m_{Q(G)}(\mu) = m_{Q(E)}(\mu)$.

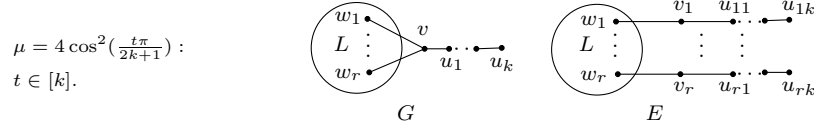


FIGURE 9. The graphs of Example 24.

5. PROOFS OF THE MAIN THEOREMS

For an $n \times m$ matrix M and $I \subseteq [n], J \subseteq [m]$, let $M[I|J]$ denote the submatrix of M formed by rows with index in I and columns with index in J .

Proof of Theorem 9. Suppose that $m_{L_{\mathcal{H}}^{\rho}}(\mu) = k$ and $\alpha^1, \dots, \alpha^k$ are the eigenvectors of $L_{\mathcal{H}}^{\rho}$ corresponding to $I_1 \dot{\cup} I_2$ by Theorem 6. We set

$$E = \left(\alpha^1 \mid \dots \mid \alpha^k \right)^T = \left(\mathbb{I}_k \mid * \right)$$

and extend α^i to $\widehat{\alpha}^i = \frac{I}{J} \begin{pmatrix} \alpha^i \\ \mathbf{0} \end{pmatrix}$, for $i \in [k]$. It is easy to check that $\widehat{\alpha}^1, \dots, \widehat{\alpha}^k$ are k μ -eigenvectors of $L_{\mathcal{E}}^{\rho}$.

Suppose that β is a μ -eigenvector of $L_{\mathcal{G}}^{\rho}$. We show that $\beta_{|I_{1j}} = \rho\beta(v_j)\mathbf{j}$, $j \in [J_1]$. We have

$$(\mu\mathbb{I} - L_{\mathcal{G}}^{\rho})_{|I}\beta_{|I} = -\rho \begin{pmatrix} X \\ 0 \\ 0 \end{pmatrix} \mathcal{A} \beta_{|J}$$

$$(3) \quad (\mu\mathbb{I} - L_{\mathcal{H}}^{\rho} - D) \begin{pmatrix} X \\ 0 \\ 0 \end{pmatrix} \mathcal{A} \beta_{|I} = -\rho \begin{pmatrix} X \\ 0 \\ 0 \end{pmatrix} \mathcal{A} \beta_{|J}.$$

Multiplying relation (3) by E from the left, we obtain

$$\mathbf{0} = E(\mu\mathbb{I} - L_{\mathcal{H}}^{\rho})\beta_{|I} = E \begin{pmatrix} D(X)\beta_{|I_1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} - \rho E \begin{pmatrix} \beta_{|J_1}X \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} - \rho E\mathcal{A}\beta_{|J_2} = \begin{pmatrix} D(X)\beta_{|I_1} - \rho\beta_{|J_1}X \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

Since, \mathbf{x}_j is nowhere-zero, hence, we have

$$(4) \quad \beta_{|I_{1j}} = \rho\beta(v_j)\mathbf{j}, \quad j \in [|J_1|].$$

Thus β is a μ -eigenvector of $L_{\mathcal{E}}^{\rho}$ by Lemma 8.

Now, we show that $m_{L_{\mathcal{E}}^{\rho}}(\mu) \geq m_{L_{\mathcal{G}}^{\rho}}(\mu) + |I_1|$. Assume that $m_{L_{\mathcal{G}}^{\rho}}(\mu) = r$ and β^1, \dots, β^r are independent eigenvectors of $L_{\mathcal{G}}^{\rho}$. We show that $\beta^1, \dots, \beta^r, \widehat{\alpha}^1, \dots, \widehat{\alpha}^{|I_1|}$ are independent. Suppose that for some $c_1, \dots, c_r, d_1, \dots, d_{|I_1|} \in \mathbb{R}$,

$$\sum_{i=1}^r c_i \beta^i = \sum_{i=1}^{|I_1|} d_i \widehat{\alpha}^i.$$

Hence $\sum_{i=1}^r c_i \beta^i = \sum_{i=1}^{|I_1|} d_i \widehat{\alpha}^i = (d_1, \dots, d_{|I_1|}, *, \dots, *)^T$.

Thus $(\mu\mathbb{I} - L_{\mathcal{G}}^{\rho})(\sum_{i=1}^{|I_1|} d_i \widehat{\alpha}^i) = \mathbf{0}$. From relation (4), $(d_1, \dots, d_{|I_1|})^T = \mathbf{0}$ and so $c_1 = \dots = c_r = 0$ and $m_{L_{\mathcal{E}}^{\rho}}(\mu) \geq m_{L_{\mathcal{G}}^{\rho}}(\mu) + |I_1|$.

Next, we show that $m_{L_{\mathcal{E}}^{\rho}}(\mu) \leq m_{L_{\mathcal{G}}^{\rho}}(\mu) + |I_1|$.

$\widehat{\alpha}^1, \dots, \widehat{\alpha}^{|I_1|}$ are $|I_1|$ μ -eigenvectors of $L_{\mathcal{E}}^\rho$. Suppose that $m_{L_{\mathcal{E}}^\rho}(\mu) = s + |I_1|$ and $\widehat{\alpha}^1, \dots, \widehat{\alpha}^{|I_1|}, \gamma^1, \dots, \gamma^s$ are independent μ -eigenvectors of $L_{\mathcal{E}}^\rho$. For $i \in [s]$, we define $\widehat{\gamma}^i$ as below,

$$\widehat{\gamma}^i = \gamma^i + \frac{I}{J_2} \left(\frac{\sum_{j \in [J_1]} E^T[I|I_{1j}](\rho\gamma^i(v_j)\mathbf{j}_{|I_1|} - \gamma_{|I_{1j}|}^i)}{\mathbf{0}} \right).$$

We show that $\widehat{\gamma}^1, \dots, \widehat{\gamma}^s$ are s independent μ -eigenvectors of $L_{\mathcal{G}}^\rho$. We have

$$\begin{aligned} L_{\mathcal{G}}^\rho \widehat{\gamma}^i &= (L_{\mathcal{E}}^\rho + \begin{pmatrix} I_1 & I \setminus I_1 & \{v\} & J_1 \\ D(X) & 0 & -\rho X & 0 \\ 0 & 0 & 0 & 0 \\ -\rho X^T & 0 & D(X^T) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix})(\gamma^i + \begin{pmatrix} \sum_{j \in [J_1]} E^T[I|I_{1j}](\rho\gamma^i(v_j)\mathbf{j}_{|I_{1j}|} - \gamma_{|I_{1j}|}^i) \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}) \\ &= \mu\gamma^i + \begin{pmatrix} \mu \sum_{j \in [J_1]} E^T[I|I_{1j}](\rho\gamma^i(v_j)\mathbf{j}_{|I_{1j}|} - \gamma_{|I_{1j}|}^i) \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} D(\mathbf{x})\gamma_{|I_1|}^i - \rho\gamma^i(v)\mathbf{x} \\ \mathbf{0} \\ -\rho\mathbf{x}^T\gamma_{|I_1|}^i + D(\mathbf{x}^T)\gamma^i(v) \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \rho\gamma^i(v)\mathbf{x} - D(\mathbf{x})\gamma_{|I_1|}^i \\ \mathbf{0} \\ -\rho^2\gamma^i(v)\mathbf{x}^T\mathbf{j} + \rho\mathbf{x}^T\gamma_{|I_1|}^i \\ \mathbf{0} \end{pmatrix} \\ &= \mu\widehat{\gamma}^i. \end{aligned}$$

Now, suppose that

$$\mathbf{0} = \sum_{i=1}^s c_i \widehat{\gamma}^i = \sum_{i=1}^s c_i \gamma^i + \begin{pmatrix} \sum_{j \in [J_1]} E^T[I|I_{1j}](\rho\gamma^i(v_j)\mathbf{j}_{|I_1|} - \gamma_{|I_{1j}|}^i) \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

for some $c_1, \dots, c_s \in \mathbb{R}$. So, $\sum_{i=1}^s c_i \gamma^i = \sum_{i \in [s], j \in [I_1]} d_{ij} \widehat{\alpha}^j$, for some real numbers d_{ij} . Thus $c_1 = \dots = c_s = 0$ and hence $m_{L_{\mathcal{E}}^\rho}(\mu) \leq m_{L_{\mathcal{G}}^\rho}(\mu) + |I_1|$. This inequality implies that $m_{L_{\mathcal{E}}^\rho}(\mu) = m_{L_{\mathcal{G}}^\rho}(\mu) + |I_1|$. \square

Proof of Theorem 13. Suppose that $m_{L_{\mathcal{H}}^\rho}(\mu) = k$ and $\alpha^1, \dots, \alpha^k$ are the eigenvectors of $L_{\mathcal{H}}^\rho$ corresponding to $I_1 \dot{\cup} I_2$ by Theorem 6. We set $E = \begin{pmatrix} \alpha^1 & \dots & \alpha^k \end{pmatrix}^T = \begin{pmatrix} \mathbb{I}_k & * \end{pmatrix}$ and extend α^i to $\widehat{\alpha}^i = \frac{I}{J} \begin{pmatrix} \alpha^i \\ \mathbf{0} \end{pmatrix}$, for $i \in [k]$. It

is easy to check that $\widehat{\alpha}^1, \dots, \widehat{\alpha}^k$ are k μ -eigenvectors of $L_{\mathcal{E}}^\rho$.

Suppose that β is a μ -eigenvector of $L_{\mathcal{G}}^\rho$. We have

$$(\mu\mathbb{I} - L_{\mathcal{G}}^\rho)|_I \beta|_I = -\rho \begin{pmatrix} X \\ 0 \\ 0 \end{pmatrix} \mathcal{A} \beta|_J$$

$$(5) \quad (\mu\mathbb{I} - L_{\mathcal{H}}^\rho - D \begin{pmatrix} X \\ 0 \\ 0 \end{pmatrix} \mathcal{A}) \beta|_I = \begin{pmatrix} L_S^\rho & 0 \\ 0 & 0 \end{pmatrix} \beta|_I - \rho \begin{pmatrix} X \\ 0 \\ 0 \end{pmatrix} \mathcal{A} \beta|_J.$$

Multiplying relation (5) by E from the left, we obtain

$$\mathbf{0} = E(\mu\mathbb{I} - L_{\mathcal{H}}^\rho) \beta|_I = E \begin{pmatrix} (L_S^\rho + D(X))\beta|_{I_1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} - \rho E \begin{pmatrix} X\beta|_{J_1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} - \rho E \mathcal{A} \beta|_{J_2} = \begin{pmatrix} (L_S^\rho + D(X))\beta|_{I_1} - \rho X\beta|_{J_1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

Hence,

$$(6) \quad \beta|_{I_1} = \rho(L_S^\rho + D(X))^{-1} X \beta|_{J_1}.$$

So, $\rho X^T \beta|_{I_1} = (L_{S'}^\rho + D(X^T)) \beta|_{J_1}$ and it is easy to see that β is a μ -eigenvector of $L_{\mathcal{E}}^\rho$.

Now, we show that $m_{L_{\mathcal{E}}^{\rho}}(\mu) \geq m_{L_{\mathcal{G}}^{\rho}}(\mu) + |I_1|$. Assume that $m_{L_{\mathcal{G}}^{\rho}}(\mu) = r$ and β^1, \dots, β^r are independent eigenvectors of $L_{\mathcal{G}}^{\rho}$. We show that $\beta^1, \dots, \beta^r, \widehat{\alpha}^1, \dots, \widehat{\alpha}^{|I_1|}$ are independent. Suppose that for some $c_1, \dots, c_r, d_1, \dots, d_{|I_1|} \in \mathbb{R}$,

$$\sum_{i=1}^r c_i \beta^i = \sum_{i=1}^{|I_1|} d_i \widehat{\alpha}^i = (d_1, \dots, d_{|I_1|}, *, \dots, *)^T.$$

Thus $(\mu \mathbb{I} - L_{\mathcal{G}}^{\rho})(\sum_{i=1}^{|I_1|} d_i \widehat{\alpha}^i) = \mathbf{0}$. From relation (6), $(d_1, \dots, d_{|I_1|})^T = \mathbf{0}$ and so $c_1 = \dots = c_r = 0$. Hence, $m_{L_{\mathcal{E}}^{\rho}}(\mu) \geq m_{L_{\mathcal{G}}^{\rho}}(\mu) + |I_1|$.

Next, we show that $m_{L_{\mathcal{E}}^{\rho}}(\mu) \leq m_{L_{\mathcal{G}}^{\rho}}(\mu) + |I_1|$.

$\widehat{\alpha}^1, \dots, \widehat{\alpha}^{|I_1|}$ are $|I_1|$ μ -eigenvectors of $L_{\mathcal{E}}^{\rho}$. Suppose that $m_{L_{\mathcal{E}}^{\rho}}(\mu) = s + |I_1|$ and $\widehat{\alpha}^1, \dots, \widehat{\alpha}^{|I_1|}, \gamma^1, \dots, \gamma^s$ are independent μ -eigenvectors of $L_{\mathcal{E}}^{\rho}$. For $i \in [s]$, we define $\widehat{\gamma}^i$ as below,

$$\widehat{\gamma}^i = \gamma^i + \frac{I}{J} \left(\frac{E^T[I|I_1](\rho(L_S^{\rho} + D(X))^{-1} X \gamma_{|J_1}^i - \gamma_{|I_1}^i)}{\mathbf{0}} \right).$$

We show that $\widehat{\gamma}^1, \dots, \widehat{\gamma}^s$ are s independent μ -eigenvectors of $L_{\mathcal{G}}^{\rho}$. We have

$$\begin{aligned} L_{\mathcal{G}}^{\rho} \widehat{\gamma}^i &= (L_{\mathcal{E}}^{\rho} + \begin{pmatrix} I_1 & I \setminus I_1 & J_1 & J_3 \\ L_S^{\rho} + D(X) & 0 & -\rho X & 0 \\ 0 & 0 & 0 & 0 \\ -\rho X^T & 0 & L_{S'}^{\rho} + D(X^T) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}) (\gamma^i + \begin{pmatrix} E^T[I|I_1](\rho(L_S^{\rho} + D(X))^{-1} X \gamma_{|J_1}^i - \gamma_{|I_1}^i) \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}) \\ &= \mu \gamma^i + \begin{pmatrix} \mu E^T[I|I_1](\rho(L_S^{\rho} + D(X))^{-1} X \gamma_{|J_1}^i - \gamma_{|I_1}^i) \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} (L_S^{\rho} + D(X)) \gamma_{|I_1} - \rho X \gamma_{|J_1} \\ \mathbf{0} \\ -\rho X^T \gamma_{|I_1} + (L_{S'}^{\rho} + D(X^T)) \gamma_{|J_1} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \rho X \gamma_{|J_1} - (L_S^{\rho} + D(X)) \gamma_{|I_1} \\ \mathbf{0} \\ -\rho^2 X^T (L_S^{\rho} + D(X))^{-1} X \gamma_{|J_1}^i + \rho X^T \gamma_{|I_1}^i \\ \mathbf{0} \end{pmatrix} \\ &= \mu \widehat{\gamma}^i. \end{aligned}$$

Now, suppose that

$$\mathbf{0} = \sum_{i=1}^s c_i \widehat{\gamma}^i = \sum_{i=1}^s c_i \gamma^i + \begin{pmatrix} E^T[I|I_1](\rho(L_S^{\rho} + D(X))^{-1} X \sum_{i=1}^s c_i \gamma_{|J_1}^i - \sum_{i=1}^s c_i \gamma_{|I_1}^i) \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

for some $c_1, \dots, c_s \in \mathbb{R}$. So, $\sum_{i=1}^s c_i \gamma^i = \sum_{i \in [s], i \in [|I_1|]} d_{ij} \widehat{\alpha}^j$, for some real numbers d_{ij} . Thus $c_1 = \dots = c_s = 0$ and hence $m_{L_{\mathcal{E}}^{\rho}}(\mu) \leq m_{L_{\mathcal{G}}^{\rho}}(\mu) + |I_1|$. This inequality implies that $m_{L_{\mathcal{E}}^{\rho}}(\mu) = m_{L_{\mathcal{G}}^{\rho}}(\mu) + |I_1|$. Finally, to prove $m_{L_{\mathcal{H}}^{\rho}}(\mu) = m_{L_{\mathcal{H}}^{\rho}}(\mu) - |I_1|$, it suffices to put $X = 0$ and $\mathcal{L} = 0$. \square

Proof of Theorem 18. We extend β^i to $\widehat{\beta}^i$ and γ^i to $\widehat{\gamma}^i$, where

$$\widehat{\beta}^i = \begin{pmatrix} I \\ I_1 \\ \vdots \\ I_{r-1} \\ J \\ J_1 \end{pmatrix} \begin{pmatrix} \beta_{|I}^i \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \beta_{|J}^i \\ \beta_{|J_1}^i \end{pmatrix}, \quad i \in [s], \quad \widehat{\gamma}^i = \begin{pmatrix} I \\ I_1 \\ \vdots \\ I_i \\ \vdots \\ I_{r-1} \\ J \\ J_1 \end{pmatrix} \begin{pmatrix} \gamma_{|I}^i \\ \mathbf{0} \\ \vdots \\ \gamma_{|I_i}^i \\ \vdots \\ \mathbf{0} \\ \gamma_{|J}^i \\ \mathbf{0} \end{pmatrix}, \quad i \in [r-1].$$

It is easy to check that $\{\widehat{\beta}^i\}_{i=1}^s$ and $\{\widehat{\gamma}^i\}_{i=1}^{r-1}$ are μ -eigenvectors of $L_{\mathcal{G}}^{\rho}$. By the definitions, the independence of these eigenvectors is obvious. \square

Proof of Theorem 21. Suppose that α is the μ -eigenvector of $L_{\mathcal{H}}^{\rho} + D(\mathbf{x})$ such that $\mathbf{x}^T \alpha = 1$ and β is a μ -eigenvector of $L_{\mathcal{G}}^{\rho}$. We have

$$(7) \quad (\mu \mathbb{I} - L_{\mathcal{G}}^{\rho})_{|I} \beta_{|I} = -\rho \begin{pmatrix} \mathbf{x} & | & 0 \end{pmatrix} \begin{pmatrix} \beta(v) \\ \beta_{|J} \end{pmatrix}$$

$$(\mu \mathbb{I} - L_{\mathcal{H}}^{\rho} - D \begin{pmatrix} \mathbf{x} & | & 0 \end{pmatrix}) \beta_{|I} = -\rho \begin{pmatrix} \mathbf{x} & | & 0 \end{pmatrix} \begin{pmatrix} \beta(v) \\ \beta_{|J} \end{pmatrix}.$$

Multiplying relation (7) by α^T from the left, we obtain

$$0 = \alpha^T (\mu \mathbb{I} - L_{\mathcal{H}}^{\rho} - D(\mathbf{x})) \beta_{|I} = -\rho \alpha^T \mathbf{x} \beta(v).$$

Hence, $\beta(v) = 0$ and $\beta_{|I} = a\alpha$ for an $a \in \mathbb{R}$. For the index v of $L_{\mathcal{G}}^{\rho}$, we have

$$(\mu - a + \rho a - D(\mathbf{x}^T) - D(\mathbf{y}^T)) \beta(v) = -\rho (\mathbf{x}^T \beta_{|I} + \mathbf{y}^T \beta_{|J}).$$

Hence,

$$(8) \quad \mathbf{x}^T \beta_{|I} = -\mathbf{y}^T \beta_{|J}, \quad a = -\frac{\mathbf{y}^T \beta_{|J}}{\mathbf{x}^T \alpha} = -\mathbf{y}^T \beta_{|J}.$$

By similar method, if γ is a μ -eigenvector of $L_{\mathcal{E}}^{\rho}$, then

$$(9) \quad \gamma(v_j) = 0, \quad \gamma_{|I_j} = a_j \alpha, \text{ and } a_j = \mathbf{x}^T \gamma_{|I_j} = -y_j \gamma(j), \text{ for } j \in [|J|].$$

Now, we show that $m_{L_{\mathcal{E}}^{\rho}}(\mu) \geq m_{L_{\mathcal{G}}^{\rho}}(\mu)$. Assume that $m_{L_{\mathcal{G}}^{\rho}}(\mu) = r$ and β^1, \dots, β^r are independent μ -eigenvectors of $L_{\mathcal{G}}^{\rho}$. Put

$$\widehat{\beta}^i = \frac{I_1 \begin{pmatrix} -y_1 \beta^i(1) \alpha \\ 0 \\ \vdots \\ -y_{|J|} \beta^i(|J|) \alpha \\ 0 \end{pmatrix}}{I_{|J|} \begin{pmatrix} -y_{|J|} \beta^i(|J|) \alpha \\ 0 \end{pmatrix}}, \quad i \in [r].$$

It is easy to see that $\widehat{\beta}^i$ is a μ -eigenvector of $L_{\mathcal{E}}^{\rho}$. We show that $\widehat{\beta}^1, \dots, \widehat{\beta}^r$ are independent. From the relation (8), we can conclude that β^1, \dots, β^r are independent, if and only if $\beta_{|J}^1, \dots, \beta_{|J}^r$ are independent. So, $\widehat{\beta}^1, \dots, \widehat{\beta}^s$ are independent and hence, $m_{L_{\mathcal{E}}^{\rho}}(\mu) \geq m_{L_{\mathcal{G}}^{\rho}}(\mu)$.

Next, we show that $m_{L_{\mathcal{E}}^{\rho}}(\mu) \leq m_{L_{\mathcal{G}}^{\rho}}(\mu)$. Suppose that $m_{L_{\mathcal{E}}^{\rho}}(\mu) = s$ and $\gamma^1, \dots, \gamma^s$ are independent μ -eigenvectors of $L_{\mathcal{E}}^{\rho}$. For $i \in [s]$, put

$$\widehat{\gamma}^i = \frac{I \begin{pmatrix} -(\mathbf{y}^T \gamma_{|J}^i) \alpha \\ 0 \\ \gamma_{|J}^i \end{pmatrix}}{v \begin{pmatrix} 0 \\ \gamma_{|J}^i \end{pmatrix}}$$

It is easy to see that $\widehat{\gamma}^i$ is a μ -eigenvector of $L_{\mathcal{G}}^{\rho}$. From the relation (9), $\gamma^1, \dots, \gamma^s$ are independent, if and only if $\gamma_{|J}^1, \dots, \gamma_{|J}^s$ are independent. So, $\widehat{\gamma}^1, \dots, \widehat{\gamma}^s$ are independent and hence $m_{L_{\mathcal{E}}^{\rho}}(\mu) \leq m_{L_{\mathcal{G}}^{\rho}}(\mu)$. This inequality implies that $m_{L_{\mathcal{E}}^{\rho}}(\mu) = m_{L_{\mathcal{G}}^{\rho}}(\mu)$. \square

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